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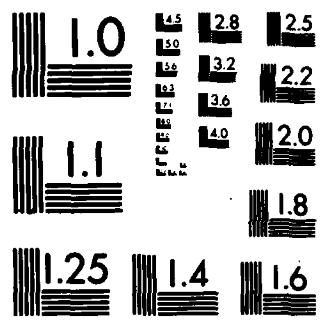

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TROPOSPHERIC - STRATOSPHERIC TIDAL INVESTIGATIONS

Notes on obtaining the eigenvalues of Laplace's tidal equation

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20. Abstract

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Notes on obtaining the eigenvalues of Laplace's tidal equation

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Abstract

A review is undertaken of the various forms that have been obtained for the recurrence relation from which the eigenvalues of Laplace's tidal equation may be obtained. Such forms are shown to be analytically consistent and are discussed in relation to their subsequent numerical evaluation. By determining eigenvalues of equivalent depth for a given frequency of oscillation instead of the other way around the problem becomes a straightforward one of matrix diagonalization. If solutions are based on normalized Legendre polynomials the matrix is symmetric A method of evaluating the related wind functions is described.

CONTENTS

1. Introduction

Laplace's tidal equation governs the latitudinal variation of the depth of a thin layer of uniform fluid on a rotating sphere that is executing small periodic oscillations in longitude ϕ and time t. Non-zero solutions are found for discrete values (eigenvalues) of the frequency of oscillation σ : alternatively, if σ is given, oscillations are found for eigenvalues of the undisturbed depth h.

Laplace's tidal equation also arises in the treatment of oscillations of a stratified and compressible atmosphere, h being replaced by a constant of separation, which is usually dimensionalized to a certain unit of length and termed an equivalent depth. In this case the equation governs the latitudinal variation of pressure (and also of temperature and vertical velocity) at any given height and h has no direct physical interpretation, being capable under certain conditions of assuming negative values. A review of the theory of atmospheric tides has been presented by Chapman and Lindzen (1970). Various extensions of the theory have since been made to damped oscillations in which case complex values of σ and h are introduced. (Semenovskiy, 1971; Volland, 1974a,b; Ishimine, 1977).

Work on the solution of Laplace's tidal equation was

pioneered by Hough (1897, 1898) in terms of associated Legendre polynomials leading to a third-order recurrence relation between the series coefficients. The derivation of this relation has been presented on several occasions and will not be repeated here: the relation is to be found in various forms which have led to different numerical procedures for evaluating the eigenvalues. This paper reviews the various theoretical results with the objective of showing their relationship and providing an understanding relevant to their numerical analysis.

2. Laplace's tidal equation

Periodic variations in longitude and time have been variously formulated in terms of either $e^{i(s\phi+\sigma t)}$ or $e^{i(s\phi-\sigma t)}$. Westward and eastward progressions of phase may then be treated by assigning both positive and negative values to one of the quantities s, σ. We limit the discussion to two forms of representation:

(i) $e^{i(s\phi+\sigma t)}$, where s = 0, ± 1, ± 2, ... and σ is a constant positive frequency, and

(ii) $e^{i(s\phi-\sigma t)}$, where s is a non-negative integer and σ a constant non-zero frequency.

Form (i) was adopted by Hough (1897) and appears more frequently in the literature than form (ii): it is found

in the review of Chapman and Lindzen (1970). Form (ii) has been adopted by Longuet-Higgins (1968), Volland (1974a) and Volland and Mayr (1977). With (i) a westward (eastward) progression of phase has s positive (negative); and with (ii) a westward (eastward) progression of phase has σ negative (positive). Any results obtained in one notation may be readily altered to apply to the other according to the scheme in Table 1.

Table 1 : Ranges of values of s, σ for westward and eastward travelling waves.

Direction of travel of wave	Form (i)	Form (ii)
Westward	$s > 0, \ \sigma > 0$	$s > 0, \ \sigma < 0$
Eastward	$s < 0, \ \sigma > 0$	$s > 0, \ \sigma > 0$

With (i), Laplace's tidal equation may be written as

$$\frac{d}{d\mu}\left[\frac{1-\mu^2}{f^2-\mu^2}\frac{d\Theta}{d\mu}\right] - \frac{1}{f^2-\mu^2}\left[\frac{s}{f}\frac{f^2+\mu^2}{f^2-\mu^2} + \frac{s^2}{1-\mu^2}\right]\Theta + \frac{4a^2\omega^2}{gh}\Theta = 0 \qquad (1)$$

where

$$\mu = \cos\theta \qquad (2)$$

$$f = \sigma/2\omega \qquad (3)$$

θ is colatitude, ω the Earth's rotation rate, a the Earth's radius and g the acceleration due to gravity. Table 1 and (3) show that with form (ii) the only change required to (1) is the replacement of s/f by -s/f in the centre term without regard to the direction of travel of the wave.

3. <u>The recurrence relation with form (i) periodic terms</u>

For given s (>0) we write

$$\Theta = \sum_{r=s}^{\infty} C_r^s P_r^s(\mu) \qquad (4)$$

Substitution of (4) into (1) leads after considerable reduction to

$$Q_{r-2} C_{r-2} + (M_r - \lambda) C_r + S_{r+2} C_{r+2} = 0 \qquad (r \geq s) \quad (5)$$

where

$$Q_{r-2} = \frac{(r-s)(r-s-1)}{(2r-1)(2r-3)\left[s/f - r(r-1)\right]} \qquad (6)$$

$$M_r = \frac{f^2\left[r(r+1) - s/f\right]}{r^2(r+1)^2} + \frac{(r+2)^2(r+s+1)(r-s+1)}{(r+1)^2(2r+3)(2r+1)\left[s/f - (r+1)(r+2)\right]}$$

$$+ \frac{(r-1)^2(r^2-s^2)}{r^2(4r^2-1)\left[s/f - r(r-1)\right]} \qquad (7)$$

$$S_{r+2} = \frac{(r+s+2)(r+s+1)}{(2r+3)(2r+5)\left[s/f-(r+1)(r+2)\right]} \tag{8}$$

$$\lambda = hg/4a^2\omega^2 \tag{9}$$

Equations (5) to (8) were first given by Hough (1898).

For given s (<0), (1) shows that Θ may be obtained by replacing s by -s (>0) and f by -f wherever they occur in (6) to (8), as (1) is unaltered by these changes of sign. For s = 0, M_0 is infinite, C_0 = 0 and Θ is based on P_2, P_4,... .

The matrix of coefficients of (5) has the form

$$M - \lambda I = 0 \tag{10}$$

where I denotes the unit matrix. On truncating M after the n th row and column, standard computer routines are available for obtaining n eigenvalues of λ . Numerical accuracy may be investigated by repeating the calculation with different values of n.

In Hough's work, fluid depth h and hence λ were taken as given and eigenvalues of f and hence σ were sought. Hough replaced (5) by two recurrence relations involving C_r and a set of auxiliary constants D_r. Then

by eliminating C_r and D_r an equation in the form of a continued fraction was obtained having an infinite number of roots for the frequencies of equatorially symmetric modes. In a like manner another continued fraction was obtained for asymmetric modes. As the numerators of the fractions were independent of f, only the denominators needed to be re-determined at each successive iteration, which was particularly advantageous for hand calculations.

If σ is given and eigenvalues for λ (and hence h) are sought, the continued fraction iteration can still be followed but appears less attractive than the matrix formulation (10) as starting values need to be taken for each solution whereas diagonalization of M yields n eigenvalues simultaneously. In recent times, eigenvalue solutions by the continued fraction method have been evaluated for symmetric modes by Kato (1966) and Ishimine (1977).

Chapman and Lindzen (1970) present equation (5) and give Hough's two sets of recurrence relations in C_r and D_r as the basis for finding eigenvalues of λ and hence h. Procedurally, the method appears less attractive than matrix diagonalization as an array size of (2n x 2n) instead of (n x n) is needed for the calculation of n eigenvalues as λ occurs only in alternate diagonal elements. Also as the form of (10) is lost, an iterative process needs to be followed to reach each root.

4. The recurrence relation with form (ii) periodic terms

Longuet-Higgins (1968), following Love (1913), introduced functions analogous to velocity potential and stream function

$$\Phi = \sum_{r=s}^{\infty} A_r P_r^s(\mu)\, e^{i(s\phi-\sigma t)} \qquad (11)$$

$$\Psi = \sum_{r=s}^{\infty} i B_r P_r^s(\mu)\, e^{i(s\phi-i\sigma t)} \qquad (12)$$

where $s \geqslant 0$. The recurrence relations

$$\begin{aligned} K_r A_r + p_{r+1} B_{r+1} + q_{r-1} B_{r-1} &= 0 \\ L_r B_r - p_{r+1} A_{r+1} - q_{r-1} A_{r-1} &= 0 \end{aligned} \qquad (13)$$

were obtained, where

$$p_r = -\frac{(r+1)(r+s)}{r(2r+1)} \qquad q_r = -\frac{r(r-s+1)}{(r+1)(2r+1)}$$

$$(14)$$

$$K_r = f + \frac{s}{r(r+1)} - \frac{r(r+1)}{(f/\lambda)} \qquad L_r = f + \frac{s}{r(r+1)}$$

The coefficient of A_{r-1} in (13) should read $-q_{r-1}$ in Longuet-Higgins (1968) instead of $-q_{r+1}$, and in equation (3.19) of that paper the sign of the term in B_n^s needs to be changed. The same equation has been reproduced in equation (28) of Volland and Mayr (1977) without correction and the same error of sign appears in equation (2.12) of

Volland (1974a) where the determinant of the coefficients in (13), namely

$$\begin{vmatrix} K_s & p_{s+1} & 0 & 0 & \cdots \\ -q_s & L_{s+1} & -p_{s+2} & 0 & \cdots \\ 0 & q_{s+1} & K_{s+2} & p_{s+3} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \end{vmatrix} \qquad (15)$$

appears without the two negative signs. On eliminating B_r from (13) we obtain

$$\frac{q_{r-2}q_{r-1}}{L_{r-1}} A_{r-2} + \left(K_r + \frac{q_{r-1}p_r}{L_{r-1}} + \frac{q_r p_{r+1}}{L_{r+1}}\right) A_r + \frac{p_{r+1}p_{r+2}}{L_{r+1}} A_{r+2} = 0 \qquad (16)$$

In the case of a uniform fluid, the dependent variable in (1) may be taken as the fluid depth and hence as the rate of change of fluid depth with time. By mass continuity the rate of decrease of fluid depth equals the divergence of fluid from a unit column, and this is proportional to $\nabla^2 \Phi$, where ∇^2 denotes the horizontal Laplacian operator on a unit sphere. Hence Θ in (1) (with s/f replaced by -s/f) is such that

$$\Theta\, e^{i(s\phi - \sigma t)} \propto \nabla^2 \Phi \qquad (17)$$

In the case of a compressible atmosphere, the proportionality is height dependent. From (4), (11) and the relation

$$\nabla^2 P_r^s = -r(r+1) P_r^s \tag{18}$$

it follows that

$$A_r \propto C_r / r(r+1) \tag{19}$$

On substituting (14) and (19) into (16) a relation in C_r is obtained which agrees with (5) on changing s/f to -s/f to accord with the use of form (ii).

The matrix of coefficients in (13) is seen by (14) to separate into terms in f and λ/f having the form (Longuet-Higgins, 1968)

$$(\lambda/f) J + C - f I \tag{20}$$

where J and C are readily constructed from (14). For given $\lambda/f = c_o$, say, eigenvalues of f may be obtained by matrix diagonalization of $c_o J + C$. This procedure does not directly solve the problem of finding eigenvalues of f for given λ, but provides a sequence of pairs of values (f_i, λ_i) such that $\lambda_i/f_i = c_o$. By taking a range of values of c_o, Longuet-Higgins (1968) was able to plot curves of $f = f(\lambda)$, each curve relating to a particular mode. Alternatively if a range of values of f were taken, the same set of curves in the form

$\lambda = \lambda(f)$ could be obtained by diagonalizing M of equation (10) (with s/f changed to -s/f). The latter procedure was followed in principle by Volland (1974a,b) in an investigation of the effect on λ (and hence on h) of varying the imaginary part of a complex f for particular modes. A matrix of coefficients in the form (20) was adopted and on noting that only alternate members of the diagonal elements K_s, L_{s+1}, K_{s+2},... contained λ, the determinant was condensed to one of half the order by operations between columns. The effect of this reduction was the same as eliminating B_r from (13) and using (16).

5. The use of normalized $P_{r,s}$

The results reviewed above have been based on the associated Legendre polynomials

$$P_r^s = (1-\mu^2)^{s/2} \frac{d^s P_r}{d\mu^s} \tag{21}$$

which are related to the normalized form $P_{r,s}$ by

$$P_r^s = \left[\frac{2(r+s)!}{(2r+1)(r-s)!}\right]^{\frac{1}{2}} P_{r,s} \tag{22}$$

Writing

$$\Theta = \sum_{r=s}^{\infty} a_r P_{r,s}(\mu) \tag{23}$$

and comparing with (4) after substituting from (22) we have

$$C_r = a_r \left[\frac{(2r+1)(r-s)!}{2(r+s)!} \right]^{\frac{1}{2}} \qquad (24)$$

Substituting (24) into (5) and multiplying through by $\left[2(r + s)!/(2r+ 1)(r - s)!\right]^{\frac{1}{2}}$, we obtain

$$L_{r-2}\, a_{r-2} + (M_r - \lambda)\, a_r + L_r\, a_{r+2} = 0 \quad (r \geqslant s) \qquad (25)$$

where

$$L_r = \frac{\left[(r+s+1)(r+s+2)(r-s+1)(r-s+2)\right]^{\frac{1}{2}}}{(2r+3)\left[(2r+1)(2r+5)\right]^{\frac{1}{2}}\left[s/f - (r+1)(r+2)\right]} \qquad (26)$$

Equations (25) and (26) have been obtained by Dikii (1965) although the factor (2r + 5) in the denominator of (26) was misprinted as (2r + s). A typographical error also appears in the recurrence relations which precede the derivation of (25), where the factor (s/f - n + 1) appearing in the coefficient of b_{n-1} should read (s/f - n - 1). Dikii (1965) obtained the expression

$$M_r = -\frac{f^2 - 1}{(s/f + r)(s/f - r - 1)} + \frac{(r-s)(r+s)(s/f - r + 1)}{(2r-1)(2r+1)(s/f + r)\left[s/f - r(r-1)\right]}$$

$$+ \frac{(r-s+1)(r+s+1)(s/f + r + 2)}{(2r+1)(2r+3)(s/f - r - 1)\left[s/f - (r+1)(r+2)\right]} \qquad (27)$$

Equation (27) may be shown to be the same relation as (7) by separating it into its component partial fractions with denominators $(s/f + r)$, $(s/f - r - 1)$, $[s/f - r(r - 1)]$ and $[s/f - (r + 1)(r + 2)]$ and then combining terms in $(s/f + r)$ and $(s/f - r - 1)$.

From (25) the matrix of coefficients is obtained as $F - \lambda I$, where

$$F = \begin{bmatrix} M_s & 0 & L_s & 0 & 0 & \cdots \\ 0 & M_{s+1} & 0 & L_{s+1} & 0 & \cdots \\ L_s & 0 & M_{s+2} & 0 & L_{s+2} & \cdots \\ 0 & L_{s+1} & 0 & M_{s+3} & 0 & \cdots \\ 0 & 0 & L_{s+2} & 0 & M_{s+4} & \cdots \\ \cdot & \cdot & \cdot & \cdot & & \end{bmatrix} \qquad (28)$$

As F is symmetric, it may be shown (Jones, 1970) that the eigenvectors are orthogonal and that, on account of the orthogonality of $P_{r,s}$, the eigenfunctions (Hough functions) are orthogonal.

Normalized spherical harmonics were adopted by Jones (1970) in the development of a general theory of atmospheric oscillations by expanding in terms of vector harmonics to obtain a matrix formulation of the equations of motion. On approximating to the case of classical tidal theory, mode decoupling is achieved by transforming to basic field variables that diagonalize a certain matrix of infinite

order whose inverse is

$$H^{-1} = L^{-1}\left[A - f^{2} T A^{-1} T\right] L^{-1} \qquad (29)$$

where T has non-zero elements

$$T_{r,r+1} = T_{r+1,r} = \left[\frac{(r+s+1)(r-s+1)r(r+2)}{(r+1)^{2}(2r+1)(2r+3)}\right]^{\frac{1}{2}} \qquad (30)$$

and

$$L = \sqrt{r(r+1)}\ 1 \qquad (31)$$

$$A = \left[1 - (s/f)/r(r+1)\right] 1 \qquad (32)$$

L and A are diagonal matrices whose diagonal elements have $r = s,\ s+1, \ldots$.

On multiplying out (29), it is found that

$$H^{-1} = f^{-2} F \qquad (33)$$

The eigenvalues of H are therefore those of F^{-1} times f^2, i.e. f^2/λ_i or $a^2\sigma^2/gh_i$ by (3) and (9).

For numerical work, the order of arrays may be halved by treating (25) as two sets of equations whose coefficients form matrices $F_0 - \lambda I$ and $F_1 - \lambda I$, where

$$F_0 = \begin{bmatrix} M_s & L_s & 0 & 0 & \cdots \\ L_s & M_{s+2} & L_{s+2} & 0 & \cdots \\ 0 & L_{s+2} & M_{s+4} & L_{s+4} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \end{bmatrix} \qquad (34)$$

$$F_1 = \begin{bmatrix} M_{s+1} & L_{s+1} & 0 & 0 & \cdots \\ L_{s+1} & M_{s+3} & L_{s+3} & 0 & \cdots \\ 0 & L_{s+3} & M_{s+5} & L_{s+5} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \end{bmatrix} \qquad (35)$$

The first set leads to eigenvectors for (a_s, a_{s+2},...) and hence to eigenfunctions by (23) that are equatorially symmetric being based on $P_{s,s}$, P_{s+2},... . The second set leads to eigenvectors for (a_{s+1}, a_{s+2},...) which are based on $P_{s+1,s}$, $P_{s+3,s}$,... and hence to the asymmetric solutions (Dikii, 1965).

6. <u>Evaluation of eigenfunctions and wind functions</u>

Before calculating Θ from (23), the vector (a_s, 0, a_{s+2}, 0,...) is normalized by dividing it by $(a_s^2 + a_{s+2}^2 + \ldots)^{1/2}$; and ($a_{s+1}$, 0, a_{s+3}, 0,...) is treated likewise. Θ is then determined apart from its sign, which is arbitrary. Signs may be chosen so that symmetric functions are positive at the equator and asymmetric functions are increasing with latitude at the equator.

The horizontal wind components of a tidal oscillation depend on horizontal gradients of the pressure field and latitudinally depend on functions derived from Θ . With form (i), these functions are (Chapman and Lindzen, 1970)

$$\Theta_U = \frac{(1-\mu^2)^{\frac{1}{2}}}{(f^2-\mu^2)^{\frac{1}{2}}}\left[\frac{s}{1-\mu^2} - \frac{\mu}{f}\frac{d}{d\mu}\right]\Theta \tag{36}$$

$$\Theta_V = \frac{(1-\mu^2)^{\frac{1}{2}}}{(f^2-\mu^2)^{\frac{1}{2}}}\left[\frac{(s/f)\mu}{1-\mu^2} - \frac{d}{d\mu}\right]\Theta \tag{37}$$

for the eastward and northward components respectively.

Series expansions of Θ_U, Θ_V in terms of $P_{r,s}$ are not obtainable, but developments are possible for

$$S_U = \sin\theta\ \Theta_U \qquad S_V = \sin\theta\ \Theta_V \tag{38}$$

From (36) and (37) we have

$$f S_U - \mu S_V - (s/f)\Theta = 0 \tag{39}$$

$$\mu S_U - f S_V - D\Theta = 0 \tag{40}$$

where $D = (1-\mu^2)d/d\mu$. For s $\geqslant$ 0, we write

$$S_U = \sum_{r=s}^{\infty} u_r P_{r,s} \qquad S_V = \sum_{r=s}^{\infty} v_r P_{r,s} \tag{41}$$

and note that (39) is unaltered for s $\geqslant$ 0 if written as

$$f S_U - \mu S_V - (|s|/f)\Theta = 0 \tag{42}$$

Then by (23), (41) and the relations

$$\mu P_{r,s} = d_r P_{r-1,s} + d_{r+1} P_{r+1,s} \tag{43}$$

$$D P_{r,s} = (r+1) d_r P_{r-1,s} - r d_{r+1} P_{r+1,s} \tag{44}$$

where

$$d_r = \left[(r^2-s^2)/(4r^2-1)\right]^{\frac{1}{2}} \tag{45}$$

the coefficients of $P_{r-1,s}$ in (40) and (42) give

$$d_r u_r = f v_{r-1} - d_{r-1} u_{r-2} - (r-2) d_{r-1} a_{r-2} + (r+1) d_r a_r \tag{46}$$

$$d_r v_r = f u_{r-1} - d_{r-1} v_{r-2} - (|s|/f)\, a_{r-1} \tag{47}$$

By successively putting r = s, s + 1,... and taking $a_r = u_r = v_r = 0$ for $r < s$, (46)and (47) enable (u_s, 0, u_{s+2}, 0,...) and (0, v_{s+1}, 0, v_{s+3},...) to be calculated for symmetric Hough functions for which $a_{s+1} = a_{s+3} = \ldots = 0$; and likewise (0, u_{s+1}, 0, u_{s+3},...) and (v_s, 0, v_{s+2}, 0,...) for asymmetric Hough functions for which $a_s = a_{s+2} = \ldots = 0$. Θ_U, Θ_V can then be obtained from (38) and (41) except for the end points $\theta = 0, \pi$.

To calculate Θ_U, Θ_V when $s < 0$, it is only necessary to replace s by -s (>0), and f by -f, as it follows from (40) and (42) that the above procedure leads

to the evaluation of Θ_U and $-\Theta_V$. If $f < 1$, (36) and (37) become indeterminate at $\mu = f$, whereas no such difficulty arises with the above procedure.

7. Discussion

The recurrence relation (5), which is that originally derived by Hough, has a matrix of coefficients of the form of equation (10) and standard computing routines are available for the determination of its eigenvalues and eigenvectors. If eigenvalues of equivalent depths are required for a given frequency of oscillation the results are obtainable in one step, but if eigenvalues of frequency are required for a given depth an approximating iteration is necessary. The latter case arises in a study of the free oscillations of the atmosphere as equivalent depth is then determined as an eigenvalue of the vertical structure: it was also the problem that concerned Hough and in devising an iterative procedure he undertook further analytical developments of the recurrence relation. These developments are not however essential to the eigenvalue analysis and it it doubtful whether they offer any advantage over the matrix formulation with present-day computing facilities.

The recurrence relation recommended for use is equation (25) which was given by Dikii and corresponds

to solutions based on normalized associated Legendre polynomials. Apart from the better numerical conditioning that is to be expected with normalization, there is the simplification of a symmetric matrix.

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